DO CLOSED UNIVERSES RECOLLAPSE?*

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ABSTRACT

It is widely believed that closed universes - those with a compact Cauchy hypersurface - behave globally as the dust-filled Friedmann universe with S^3 spatial topology: start at an all-encompassing initial singularity, expand to a maximal hypersurface, and recollapse to an all-encompassing final singularity. In reality, it is not known if the generic S^3 closed universe recollapses. In fact, I shall show that there are even S^3 *Friedmann* universes satisfying all the standard energy conditions (and with zero cosmological constant) that expand forever. However, if a generic closed universe at some point in its history attains a maximal hypersurface, then it does originate at an initial all-encompassing initial singularity, and does recollapse to an all-encompassing final singularity. But only certain spatial topologies admit maximal hypersurfaces, and hence permit recollapse: roughly speaking, the only closed universes which can ever evolve maximal hypersurfaces are those whose Cauchy hypersurfaces have topology S^3 or $S^2 \times S^1$, or a more complicated topology formed from these two basic types by connected summation and certain identifications. All known solutions to Einstein's vacuum equations with S^3 or $S^2 \times S^1$ Cauchy hypersurface topology recollapse, so I conjecture that *all* vacuum solutions with these Cauchy hypersurface topologies recollapse. I shall also state a recollapse conjecture for matter-filled spatially homogeneous closed universes, and give a general recollapse theorem for Friedmann universes: if the positive pressure criterion, the dominant energy condition and the matter regularity condition hold, then an S^3 Friedmann universe originates at an initial singularity, expands to a maximal hypersurface, and recollapses to a final singularity. Counter-examples indicate that this Friedmann recollapse theorem is more or less the most general recollapse theorem for the Friedmann universe.

1. WHO CARES IF CLOSED UNIVERSES RECOLLAPSE?

Since this Symposium on Relativistic Astrophysics consists more of astrophysicists than relativists, I should like to provide a justification for investigating the Recollapse Problem to the former, who generally think the business of science is finding an explanation of observed past or present phenonmena, and who often think that the behavior of the universe in the far future is therefore irrelevant to science in general and to their work in particular. There are at least three reasons for regarding the Recollapse problem as important to physical cosmology.

The first is rather trivial: at some point in her career, every astrophysicist teaches an elementary astronomy course, including some cosmology. The elementary texts almost uniformly assert that closed universes, by which the texts usually mean universes with S^3 spatial topology, recollapse. In fact, it is not known if "realistic" S^3 closed

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universes recollapse, where "realistic" means a universe which is inhomogeneous and which contains the matter fields of contemporary particle physics. Furthermore, only closed universes with certain very special spatial topologies can have a maximal hypersurface and hence recollapse: in particular, a closed universe with a T^3 spatial topology cannot admit a maximal hypersurface and hence (probably) must expand forever.

The second reason is that the early universe behavior of many quantum cosmological models depend on their global temporal structure. For example, the Hartle-Hawking model[1] in its simplest form postulates that the wave function of the universe is defined globally on S^4, and only when the densities are significantly below the Planck density -- that is, far away from what classically would be the initial and final singularities -- can the universe be described as topologically $S^3 \times R^1$, and metrically spacetime. The reason for limiting the domain of the wave function to a compact 4-manifold is to eliminate the necessity for global boundary conditions; as Hawking puts it, the most plausible boundary condition is that there is no boundary. But this compactness in 4 dimensions requires recollapse, for an ever-expanding universe is open in the future temporal direction, and thus would require boundary conditions at future temporal infinity. In a sense, these future boundary conditions are avoided in the Hartle-Hawking model by identifying the high density future with the high density past (though "past" and "future" lose their meaning in these high density regions[1], in part because it is no longer meaningful to talk about the trajectory of a single classical universe); the early universe depends on the future because quantum mechanically (but not classically - see Hawking's paper in these Proceedings), the past *is* the future. This identification can be carried out only if we have recollapse. To the extent we are interested in whether the Hartle-Hawking model correctly describes the early universe, we are interested in the Recollapse Problem.

The third reason is that the plausibility of the inflationary model of the universe depends in part on whether closed universes recollapse. The inflationary model derives its appeal by purporting to show that certain major features of the visible universe (the fact that it is nearly flat, for example) are nearly independent of the initial conditions. However, inflationary models will in fact be nearly independent of the initial conditions only if they asymptotically approach the de Sitter state during the accelerating phase. Whether or not this approach occurs is termed the "cosmic no hair conjecture". As pointed out by Barrow[2,3], attempts to prove no-hair theorems have assumed that the spatial 3-curvature scalar is non-positive on the grounds that universes with positive 3-curvature scalar recollapse. But in fact, I shall give below S^3 Friedmann models (all S^3 Friedmann models necessarily have positive 3-curvature scalar) which expand forever. On the other hand, if more realistic S^3 closed models *do* recollapse, it is possible that *generically* they recollapse too soon for inflation to occcur, suggesting that inflation is unlikely to occur closed universes.

My conventions will be those of Hawking and Ellis[4]. The cosmological constant will be assumed to be zero. I shall in large part be summarizing work done jointly by myself and J.D. Barrow[5] and by myself, J.D. Barrow and G.J. Galloway[6].

2. IMPLICATIONS OF MAXIMAL HYPERSURFACES

The first theorem establishes the necessity of all-encompassing initial and final singularities in a universe with a compact maximal hypersurface. Recall that a spacelike hypersurface S is said to be a *maximal hypersurface* if $z^a{}_{;a} = 0$ everywhere on S, where z^a is the unit normal to S. A singularity to the past of a spacetime point set **P** will be said to be *all-encompassing* if every inextendible timelike curve λ in $I^-(\mathbf{P})$ - i.e., in the past of

P - has a proper time length less than a universal constant L (i.e., the length of $\lambda \cap I^-(\mathbf{P})$ is less than L). An all-encompassing final singularity is defined analogously.

Theorem 1: Let S be a compact maximal Cauchy hypersurface. Then there is an all-encompassing singularity to the past of S and an all-encompassing singularity to the future of S, and further the length of *every* timelike curve in the entire spacetime is less than a universal constant L, provided

(1) $R_{ab}V^aV^b \geq 0$, for all timelike vectors V^a;

(2) At least one of the tensors $z^c z^d z_{[a}R_{b]cd[e}z_{f]}$, $z_{a;b}$, or $R_{ab}z^a z^b$, is non-zero on S, where z^a is the normal vector to S.

Theorem 1 was first stated and proved by Marsden and Tipler[7]. Condition (1), the timelike convergence condition, merely says that gravity is always attractive. Condition (2) says that somewhere on the maximal hypersurface, the gravitational tidal forces are non-zero, or at least the hypersurface is not a hypersurface of time symmetry. For vacuum spacetimes, a hypersurface S of time symmetry ($z_{a;b} = 0$ everywhere on S) would imply that the future and past of S are identical. It is very unlikely that the gravitational tidal forces are identically zero everywhere on S, so condition (2) is a generic condition.

The next theorem shows that a maximal hypersurface will never evolve in some universes with compact Cauchy hypersurfaces; only certain topologies admit maximal hypersurfaces.

Theorem 2: If S is a spacelike compact orientable maximal hpersurface, then it must have the topology

$$[S^3]_1 \# [S^3]_2 \#...\# [S^3]_n \# k(S^2 \times S^1)$$

(where $[S^3]_i$ is a manifold which is covered by a homotopy 3-sphere, "#" denotes the connected sum, and $k(S^2 \times S^1)$ means the connected sum of k copies of $S^2 \times S^1$), provided the following hold:

(1) The Einstein equations without cosmological constant hold,
(2) the weak energy condition holds, and
(3) the induced metric on S is not flat.

In particular, since T^3 cannot be so written, a closed universe whose Cauchy hypersurface has topology T^3 cannot evolve a maximal hypersurface. Theorem 2 was first proved in [5] (see also [8]). The Theorem is an application of a theorem of Schoen and Yau[9], later generalized by Gromov and Lawson[10]. Witt[11] has recently applied the Schoen-Yau theorem to the existence of maximal hypersurfaces in asymptotically flat space. The hypotheses and conclusions in Theorem 2 are weaker that those of the equivalent theorem in [5] and [8] : in the latter, the manifold $[S^3]_i$ is S^3/P_i, the quotient of S^3 with P_i , a subgroup of O(4) which acts standardly on S^3. To obtain this stronger conclusion, an additional hypothesis ruling out exotic differentiable structures was made; in effect this added hypothesis ruled out homotopy spheres which are not spheres (i.e., it explicitly ruled out manifolds which violate the Poincaré Conjecture), and it ruled out more exotic identifications of S^3 than S^3/P_i. (I am grateful to J. Friedman for discussions on this point.) Schoen and Yau[9] give other hypotheses which reduce $[S^3]_i$ to

S^3/P_i. The important point is that the only recollapsing universes with*simple* spatial topologies are those with either S^3 or $S^2 \times S^1$ spatial topology.

3. RECOLLAPSE IN S^3 FRIEDMANN UNIVERSES

The archetypical S^3 closed universe is the closed Friedmann universe, so it is interesting that there are S^3 Friedmann models that expand forever but in which the matter obeys all the standard energy conditions. To see this, recall that if the matter is a perfect fluid with equation of state $p = (\gamma - 1)\mu$, conservation of energy implies that $\mu = (3M/8\pi G)R^{-3\gamma}$, where M is a constant and R is the usual Friedmann scale factor. The Friedmann constraint equation is thus

$$(R'/R)^2 = M/R^{3\gamma} - 1/R^2 \qquad (1)$$

The homogeneity, isotropy, and S^3 spatial topology imply $\mu > 0$ and $M > 0$. Clearly the universe expands forever if $\gamma \leq 2/3$ (if $\gamma = 2/3$, $M > 1$, since the LHS of (1) is positive if the universe is expanding initially). It is easily checked that if $\gamma = 2/3$, the weak, strong, and dominant energy conditions are satisfied.

If $\gamma = 2/3$, the generic condition is not satisfied, but we can add dust satisfying $\mu = (3M_D/8\pi G)R^{-3}$ to the fluid having $\gamma = 2/3$, with M and M_D chosen so that the Friedmann constraint equation is

$$(R'/R)^2 = - M_D/R + 1 \qquad (2)$$

which is the Friedmann equation for dust, but with $k = -1$ (the open Friedmann universe); clearly such an S^3 Friedmann universe expands forever, and it is easily checked that the generic condition is satisfied, together with all the other above mentioned energy conditions. What happens is this: when $\gamma < 1$, the pressure is negative, and in general relativity, negative pressure generates a *repulsive* gravitational force (this is why the inflationary universe inflates; some of the fields now being considered by particle physicists have strong negative pressures[6]). When $\gamma \leq 2/3$, this repulsion overwhelms the attractive force due to $\mu > 0$; i.e., the attractive force due to the positive spatial curvature.

If negative pressures are eliminated, we can prove S^3 Friedmann universes recollapse:

Theorem 3: If the positive pressure criterion, the matter regularity condition, and the dominant energy condition hold, then a Friedmann universe with S^3 spatial topology expands from an initial singularity to a maximal hypersurface, and then recollapses to a final singularity.

Theorem 3 is proved in [6]. Counter-examples[6] indicate that the hypotheses of Theorem 3 cannot be significantly weakened. The *positive pressure critierion* of Collins and Hawking[12] says that $\Sigma p_i \geq 0$, where p_i are the principal pressures of the stress-energy tensor. In the Friedmann universe, the 3 principal pressures are all equal, so the criterion reduces to $p \geq 0$. In more general spacetimes, the positive pressure criterion is much weaker than $p_i \geq 0$, $i = 1,2,3$; in fact, the latter condition is violated in certain Bianchi type IX S^3 universes containing electromagnetic fields[6], but the postive pressure criterion holds, and recollapse occurs. The *matter regularity condition*[6] -- which, roughly speaking, asserts that the stress-energy tensor is well-behaved except at a p.p. curvature singularity[4] -- and the dominant energy condition are required to ensure that the pressure doesn't blow up and stop the evolution before the maximal hypersurface is reached. (Don't laugh -- this can actually happen in S^3 Friedmann universes[6].)

4. CLOSED UNIVERSE RECOLLAPSE CONJECTURES

All known vacuum solutions to Einstein's equations with Cauchy hypersurface topology S^3 or $S^2 \times S^1$ are known[6] to recollapse, so I propose

Conjecture 1: All globally hyperbolic vacuum C^2 maximally extended closed universes with S^3 or $S^2 \times S^1$ spatial topology expand from an all-encompassing initial singularity to a maximal hypersurface, and recollapse to an all-encompassing final singularity.

Examples indicate[6] that the conditions on the matter tensor in Theorem 3 are sufficient to obtain recollapse, at least in homogeneous universes, so I therefore propose

Conjecture 2: All globally hyperbolic C^2 maximally extended spatially homogeneous closed universes with S^3 or $S^2 \times S^1$ spatial topology, and with stress-energy tensors which obey

(1) the strong energy condition,
(2) the positive pressure criterion,
(3) the dominant energy condition, and
(4) the matter regularity condition,

expand from an all-encompassing initial singularity to a maximal hypersurface, and recollapse to an all-encompassing final singularity.

I challenge the reader to prove these conjectures, or to give counter-examples.

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